NUWC-NPT Technical Report 10,793
1 December 1994

Approximation of the Page Test Probability of Detection by the Cumulative Distribution of a Mixture of Poisson Random Variables

Douglas A. Abraham
Submarine Sonar Department

Naval Undersea Warfare Center Division
Newport, Rhode Island

PREFACE

The research described in this report was funded under NUWC Detachment New London Project No. B10001, *Quickest Detection of Signals in the Presence of Noise With Unknown Parameters,* principal investigator D. A. Abraham (Code 2121). The work was sponsored by the Independent Research Program, program manager K. M. Lima (Code 102).

The technical reviewer for this report was A. C. Barthelemy (Code 2151).

Reviewed and Approved: 1 December 1994

R. J. Martin
Acting Head, Submarine Sonar Department

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (Leave Blank)

2. REPORT DATE
1 December 1994

3. REPORT TYPE AND DATES COVERED
Final

4. TITLE AND SUBTITLE
Approximation of the Page Test Probability of Detection by the Cumulative Distribution of a Mixture of Poisson Random Variables

5. FUNDING NUMBERS

6. AUTHOR(S)
Douglas A. Abraham

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Undersea Warfare Center
Detachment New London
New London, Connecticut 06320

8. PERFORMING ORGANIZATION REPORT NUMBER
TR 10,793

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

Sequential tests like the Page test have been suggested for the detection of finite, unknown duration signals occurring at unknown times. The standard performance measures for the Page test are the average number of samples between false alarms and the average number of samples before detection. The average number of samples between false alarms is an appropriate false alarm performance measure as the time of occurrence of the signal is unknown. However, the probability of detection is a more appropriate performance measure than the average number of samples before detection for signals having finite duration. Unfortunately, there has been minimal research related to determining this performance measure for the Page test. The results of Han, Willett, and Abraham indicate that the probability of detecting a finite duration signal with the Page test may be accurately approximated through the use of a continuous-time Brownian motion model or a quantized continuous-time process with moment matching when the sequence submitted to the regulated cumulative summation in the Page test is Gaussian. As many problems of interest result in non-Gaussian sequences to be submitted to the Page test, it is desired to obtain distribution-independent methods for accurate detection probability approximation. The method proposed and analyzed herein involves approximating the probability density function of the stopping time random variable (i.e., the number of samples before detection) by a mixture of Poisson random variables. The probability of detecting a signal as a function of its duration is then approximated by the cumulative distribution function of the Poisson mixture.

14. SUBJECT TERMS
Cumulative Distribution Function, False Alarm Performance, Page Test, Poisson Random Variable, Probability of Detection, Sequential Test, Signal Detection, Signal Duration

15. NUMBER OF PAGES
34

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	**UNCLASSIFIED**	**UNCLASSIFIED**	**SAR**

NSN 7540-01-280-5500

Standard Form 298 (Rev 2-89)
Prescribed by ANSI Std Z39-18
298-102

TABLE OF CONTENTS

Page

LIST OF ILLUSTRATIONS

LIST OF TABLES

LIST OF SYMBOLS

$\bar{T}$	Average number of samples between false alarms
$\bar{D}$	Average number of samples before detection
$P_d(L)$	Probability of detecting a signal of duration L
ϕ_i	Mean parameter for Poisson or geometric random variable
α_i	Proportion parameter for Poisson mixture
Φ	Vector of mean and proportion parameters
n	Number of data observations
m	Number of components in mixture model
Δ_{Par}	Change in parameter estimates during EM update
Δ_{LLF}	Change in log-likelihood function during EM update
TOL_1	Stopping tolerance on change in log-likelihood function
TOL_2	Stopping tolerance on change in parameter estimates
TOL_3	Stopping tolerance on change in log-likelihood function for model estimation
X_{max}	Maximum value to include error between estimated and actual CDF
$r(z)$	Locally optimal nonlinearity
τ_s	Asymptotically optimal bias
s	Signal strength or signal-to-noise ratio
W_k	Page test statistic
$\lfloor \cdot \rfloor$	Floor function

LIST OF ABBREVIATIONS AND ACRONYMS

AIC	Akaike's information criterion
CDF	Cumulative distribution function
EM	Estimate-maximize
LLF	Log-likelihood function
MDL	Minimum descriptive length
MLE	Maximum likelihood estimate
MSE	Mean squared error
PDF	Probability density function
SNR	Signal-to-noise ratio

APPROXIMATION OF THE PAGE TEST PROBABILITY OF DETECTION BY THE CUMULATIVE DISTRIBUTION OF A MIXTURE OF POISSON RANDOM VARIABLES

INTRODUCTION

The detection of finite, unknown duration signals that occur at unknown times has often been a problem in the fields of active and passive sonar signal processing. Examples of signals requiring such detection include those caused by an active sonar return in a shallow water environment and by the onset of a range bias in target motion analysis. Also important is the capability for the *rapid* detection of a torpedo or ship closing in range. It has been suggested ([1], [2]) that sequential detectors, such as the Page test [3], be used. This report investigates a technique for approximating the probability of detecting signals using the Page test as a function of their duration and strength.

The Page test is implemented by the update equation

$$W_k = \max\{0, W_{k-1} + Y_k\}, \tag{1}$$

where $W_0 = 0$. The sequence $\{Y_k\}$ is the Page test update, which is optimally the log-likelihood ratio of the observed data for time sample k. However, this update may be any function of the data whose mean is negative when no signal is present. The stopping time of a sequential test is defined as the first time or index when the stochastic test statistic process or sequence crosses a given threshold, indicating a detection, false or otherwise. Thus, the stopping time for the Page test is defined as

$$N = \inf\{k \geq 0 : W_k \geq h\}, \tag{2}$$

where h is the threshold. The Page test is actually designed to rapidly determine when a sequence permanently changes from following one distribution law to an alternative law. Thus, its traditional performance measures are the average number of samples between false alarms denoted, $\bar{T}$, and before detection, $\bar{D}$. From a sonar operator's perspective, $\bar{T}$ is an appropriate false alarm performance measure as it relates to how often action is falsely

required. For finite duration signals, the probability of detection as a function of signal duration and strength is a more appropriate detection performance measure than $\bar{D}$.

Theoretical results bounding or approximating the probability of detecting a finite duration signal with the Page test may be found in Broder [1] and Han, Willett, and Abraham [4]. Broder discusses approximating the Page test probability of detection by exploiting the asymptotic normality of the stopping time (as established by Khan [5]), by deriving a lower bound based on the performance of a fixed sample size test, and by approximating the Page test sequence by use of an *unregulated* continuous-time Brownian motion [6]. In reference [4], the Page test statistic sequence was approximated by a regulated, continuous-time Brownian motion, or by a quantized continuous-time process where the first two moments are matched to those of the Page test. The characteristic function of the stopping time of the continuous-time processes may then be determined analytically. The cumulative distribution function (CDF) of the stopping time, obtained numerically from the characteristic function, provides an approximation to the Page test probability of detection. The results of reference [4] indicate that both methods are accurate when the Page test update has a Gaussian random character and that the moment-matching method may provide adequate estimation for non-Gaussian updates.

Using analyzable alternatives to model the statistical character of the stopping time random variable provides good results. This approach is practical because if the probability density function (PDF) of the stopping time random variable is available, the resulting CDF provides a lower bound on the probability of detecting a signal using the Page test:

$$P_d(L) \geq \Pr\{N \leq L \,|\text{Signal present}\}. \tag{3}$$

Here, N is the stopping time random variable of equation (2) and L is the duration of the signal. This CDF is a lower bound because a valid detection may occur in the Page test after the signal has stopped due to memory in the test statistic and because it is typically assumed that the Page test statistic is zero when the signal starts, which is the worst possible value.

Many problems of interest, particularly those with unknown noise or "nuisance" parameters, result in statistics that may be distinctly non-Gaussian. Thus, there is a need for determining the Page test probability of detection when the Page test update is non-Gaussian.

The results of reference [4] may be used as a first order approximation, depending on how well a Gaussian random variable approximates the non-Gaussian Page test update. An alternative approach is to estimate the CDF of the stopping time random variable through simulation. Accurate estimation, however, may require substantial amounts of data. The technique proposed herein is to approximate the PDF of the stopping time by a mixture of Poisson random variables. The CDF of the observed data is then estimated by the CDF of the Poisson mixture. The estimate-maximize (EM) algorithm [7] is used to obtain maximum likelihood estimates (MLEs) of the proportion and mean parameters for the individual Poisson random variables. This estimator for the CDF of a non-negative integer-valued random variable is described, analyzed, and compared to the sample CDF. The Poisson mixture method, the Brownian motion and moment-matching approximations of [4], and the fixed sample size and asymptotic approximations of [1] are then compared for approximating the Page test probability of detection for several common signal types.

POISSON MIXTURE APPROXIMATION

Suppose that n observations of a non-negative integer-valued random variable distributed according to the PDF $f(x)$ are observed:

$$x_i \sim f(x); \quad x \in \{0, 1, 2, \ldots\}, \quad i = 1, 2, \ldots, n\,. \tag{4}$$

It is desired to estimate the CDF of the observed data,

$$F(x) = \sum_{j=0}^{x} f(j)\,, \tag{5}$$

by approximating the PDF by a mixture of Poisson random variables. A Poisson random variable is parameterized by its mean value, λ, and has PDF

$$p(x|\lambda) = \frac{\mathrm{e}^{-\lambda}\lambda^{x}}{x!}, \quad \text{for} \quad x = 0, 1, 2, \ldots\,. \tag{6}$$

A mixture of m Poisson random variables with means ϕ_i and mixture proportions α_i has a PDF with the form

$$g(x|\phi, \alpha) = \sum_{i=1}^{m} \alpha_i p(x|\phi_i)\,, \tag{7}$$

where

$$\sum_{i=1}^{m} \alpha_i = 1 \tag{8}$$

and ϕ and α are length m vectors of the mean and proportion parameters, respectively. Because of the constraint of equation (8), the dimension of the parameter space is $2m - 1$.

Application of the EM algorithm [7] to determine the MLEs of the mixture parameters is straightforward, as the Poisson distribution is a member of the exponential family. The following update equations of the EM iteration for the Poisson mixture may be found in Redner and Walker [8]. Let the current estimates of the proportion and mean parameters be α_i^c and ϕ_i^c, respectively, for $i = 1, \ldots, m$. Define the intermediate variables

$$\begin{aligned} w_{i,k} &= \alpha_i^c p(x_k|\phi_i^c) \\ &= \frac{\alpha_i^c \mathrm{e}^{-\phi_i^c}(\phi_i^c)^{x_k}}{x_k!}, \end{aligned} \tag{9}$$

for $i = 1, \ldots, m$, and for $k = 1, \ldots, n$. The update for the mean parameters is

$$\phi_i^+ = \frac{\sum_{k=1}^{n} x_k \frac{w_{i,k}}{\sum_{j=1}^{m} w_{j,k}}}{\sum_{k=1}^{n} \frac{w_{i,k}}{\sum_{j=1}^{m} w_{j,k}}}, \tag{10}$$

and the update for the proportion parameters is

$$\alpha_i^+ = \frac{1}{n} \sum_{k=1}^{n} \left(\frac{w_{i,k}}{\sum_{j=1}^{m} w_{j,k}} \right) \tag{11}$$

for $i = 1, \ldots, m$. Note that the constraint of equation (8) is satisfied by the update equation for the mixture proportion parameters.

Equations (9)–(11) describe how to update estimates of the mixture parameters given a specified model order and an initial estimate of the parameters. In the following three sections, the initialization, stopping, and model order choice are discussed.

INITIALIZATION

There is little discussion in Redner and Walker [8] about initialization of the EM algorithm for mixtures of exponential family densities, except to note a linear (slow) convergence from *most all* points to a local maxima of the likelihood function. It is assumed that the mean parameters of the individual Poisson random variables in the order m mixture are unique and may be ranked in ascending order. Thus, a natural initial estimate of the i^{th} ranked mean parameter from n samples of unlabeled data is the average of the i^{th} set of $\lfloor \frac{n}{m} \rfloor$ ranked data observations, where $\lfloor \cdot \rfloor$ is the floor function. The mixture proportions are initialized to equal values of $\alpha_i = \frac{1}{m}$ for $i = 1, \ldots, m$. This initialization procedure was suggested by T. E. Luginbuhl (Code 2121) of the Naval Undersea Warfare Center Detachment in New London, Connecticut.

ALGORITHM STOPPING

The objective of the EM algorithm is to determine parameter estimates maximizing the Poisson mixture PDF. Thus, a combination of the change in the likelihood function (the joint PDF) of the data and a distance measure between successive parameter vector estimates is used as a stopping criterion. Let the superscripts ^{c} and $^+$ on the vectors ϕ and α,

respectively, indicate the old and new parameter estimates. The distance measure between successive parameter estimates is

$$\Delta_{Par} = \frac{\|\alpha^{+} - \alpha^{c}\|_2}{1 + \|\alpha^{+}\|_2} + \frac{\|\phi^{+} - \phi^{c}\|_2}{1 + \|\phi^{+}\|_2}, \tag{12}$$

where $\|\mathbf{x}\|_2 = \left(\mathbf{x}^T\mathbf{x}\right)^{\frac{1}{2}}$ is the 2-norm of a vector. The change in the log-likelihood function (LLF) was measured by

$$\Delta_{LLF} = \frac{\log g\left(\mathbf{x}|\phi^{+}, \alpha^{+}\right) - \log g\left(\mathbf{x}|\phi^{c}, \alpha^{c}\right)}{1 + \left|\log g\left(\mathbf{x}|\phi^{+}, \alpha^{+}\right)\right|}, \tag{13}$$

where $\mathbf{x}$ is the n-by-1 vector of the observed data and

$$g\left(\mathbf{x}|\phi, \alpha\right) = \prod_{k=1}^{n} g\left(x_k|\phi, \alpha\right) \tag{14}$$

is the joint PDF of the observed data under the mixture model parameterized by ϕ and α. The EM iteration is halted when the change measures are less than a specified tolerance:

$$\Delta_{LLF} < TOL_1 \tag{15}$$

and

$$\Delta_{Par} < TOL_2 \,. \tag{16}$$

MODEL DIMENSION CHOICE

The above EM algorithm initialization and implementation requires that the dimension of the mixture model be specified. As this is assumed to be unknown, Akaike's information criterion (AIC) [9] and Rissanen's minimum descriptive length (MDL) [10] are used to determine the best model dimension. As there are $2m - 1$ model parameters, the AIC and MDL objective functions to be minimized are

$$\mathrm{AIC}\,(m) = -\log\left[g\left(\mathbf{x}|\hat{\phi}_m\right), \hat{\alpha}_m\right] + 2m \tag{17}$$

and

$$\mathrm{MDL}\,(m) = -\log\left[g\left(\mathbf{x}|\hat{\phi}_m, \hat{\alpha}_m\right)\right] + m\log n \,, \tag{18}$$

where $g(\mathbf{x} \mid \phi, \alpha)$ is as shown in equation (14), n is the number of data observations, and $\hat{\phi}_m$ and $\hat{\alpha}_m$ are the MLEs of the mixture parameter vectors when there are m mixture components.

Note that these functions only depend on the mixture parameter estimates through the LLF. It has been observed that Δ_{LLF} converges rapidly, whereas Δ_{Par} may take a substantial amount of iterations for convergence. Thus, the model dimension is determined by estimating the mixture parameters for $m = 2, 3, \ldots$ components, stopping the EM iteration each time when

$$\Delta_{LLF} < TOL_3 \,, \tag{19}$$

until equation (17) or (18) produces a minimum. The mixture parameter estimates resulting from the best model dimension are then used to continue the EM iteration until $\Delta_{LLF} < TOL_1$ and $\Delta_{Par} < TOL_2$.

PERFORMANCE ANALYSIS

The primary objective in approximating the PDF of the stopping time random variable by use of a mixture of Poisson random variables is to approximate the Page test probability of detection by the CDF of the Poisson mixture. Thus, it is worthwhile to explore the performance of the above described algorithm for estimating a CDF. The fundamental questions are

1. How much observed data are required to obtain an accurate estimation?
2. Should the AIC or the MDL criteria be used to choose the model order?
3. How does the performance compare to the sample CDF estimator and MLEs for a mixture model with the correct order?
4. How well does the algorithm work when the underlying distribution is not a mixture of Poisson random variables?

To address these questions, a Poisson mixture example and a Poisson-geometric mixture example are considered. In example 1, the observed data are distributed as a mixture of five

Poisson random variables with proportion and mean parameters as found in table 1. Example 2 consists of a mixture of a Poisson and two geometric random variables, as described in table 2. For the geometric distribution, ϕ_i is equal to $\frac{1}{p_i}$, where p_i is the probability of stopping at the next sample. Sample CDF estimates and the mean-squared error (MSE) between the estimated CDF and the actual CDF for these examples are found in figures 1–4.

Table 1: Proportion and Mean Parameters for Example 1

α_i	ϕ_i
0.2	3
0.1	5
0.2	8
0.4	10
0.1	20

Table 2: Proportion and Mean Parameters for Example 2

Type	α_i	ϕ_i
Poisson	0.2	5
Geometric	0.4	5
Geometric	0.4	10

The actual CDF, the sample CDF, and the mixture-model-based AIC CDF estimate (model order chosen according to the AIC objective function) are found in figures 1 and 3 for each example, where $n = 50$ data observations is used in example 1 and $n = 200$ data observations is used in example 2. The AIC criteria chose a model order of $m = 3$ for each example, underestimating the actual model order in example 1. This may be attributed to the clumping of the actual mean parameters into three groups centered at 4, 9, and 20. These plots illustrate that both the sample CDF and the mixture-model-based CDF estimate only approximate the actual CDF.

Of more interest is the quality of the approximation as the amount of data observations varies. The MSE between the estimated CDF and the actual CDF is used as a performance measure:

$$\mathrm{MSE}\left(X_{\max}\right) = \mathrm{E}\left[\sum_{x=0}^{X_{\max}} \left(F\left(x\left|\hat{\phi},\hat{\alpha}\right.\right) - F\left(x\right)\right)^2\right], \tag{20}$$

where $F(x)$ is the CDF of the observed data and

$$F\left(x\left|\phi,\alpha\right.\right) = \sum_{k=0}^{x} g\left(x\left|\phi,\alpha\right.\right) \tag{21}$$

is the Poisson mixture CDF given the parameters in ϕ and α. Here, the expectation occurs over the random estimate of the mixture parameters $\hat{\phi}$ and $\hat{\alpha}$, and $X_{\max}$ is taken large enough so that the probability of observing a value exceeding $X_{\max}$ is nearly zero.

The MSE measure of equation (20) is estimated by forming the sample mean of the squared error (the term within the expectation in equation (20)) over 100 trials. The results are shown in figure 2 for example 1, where $X_{\max} = 50$, and in figure 4 for example 2, where $X_{\max} = 100$, for the sample CDF, the mixture model with the AIC and MDL criteria for model order choice, and the mixture model with the correct number of components (example 1 only). Note that the methods utilizing the AIC and MDL criteria achieve nearly the same MSE as the Poisson mixture method with the correct model order in example 1. Upon closer examination, it is observed that the AIC criteria yields slightly better performance in both examples. However, as seen in figures 5 and 6, this result is not statistically significant because the MDL method lies within the three-standard-deviation confidence region of the estimated mean for the AIC method. For the values of n considered, the sample CDF estimator has an MSE more than one order of magnitude greater than the mixture-model-based estimators.

Comparison of the performance of the mixture-model-based CDF estimators for the correctly modeled Poisson mixture and the incorrectly modeled Poisson-geometric mixture shows that a larger number of data observations are required to obtain equivalent accuracy for the incorrectly modeled data. As this is also true for the sample CDF estimator, it is not clear if this result is due to the algorithm or to the Poisson-geometric mixture of example 2. However, adequate estimation is achievable, even for distributions that are not Poisson mixtures, as long as the amount of data used to estimate the mixture parameters is large enough (e.g., $n = 500$).

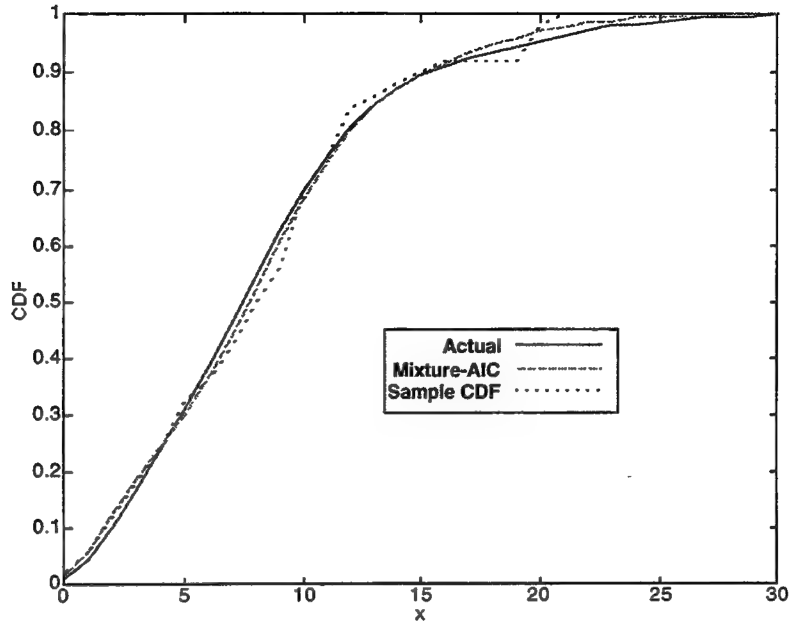

Figure 1: Actual CDF, Sample CDF, and Mixture-Model-Based AIC CDF Estimate of Example 1 With $n = 50$ Observations

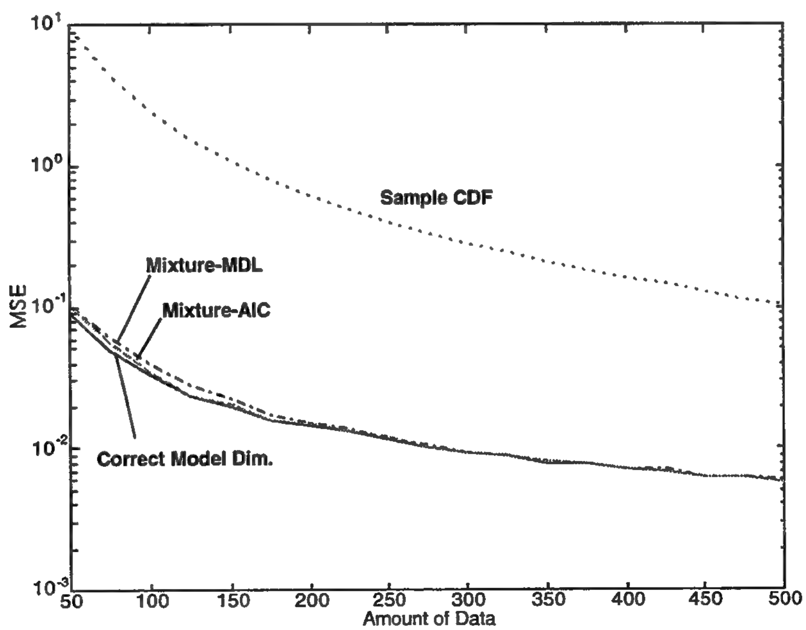

Figure 2: MSE for Sample CDF and Mixture-Model-Based CDF Estimate Using AIC, MDL, and Actual Model Order for Example 1

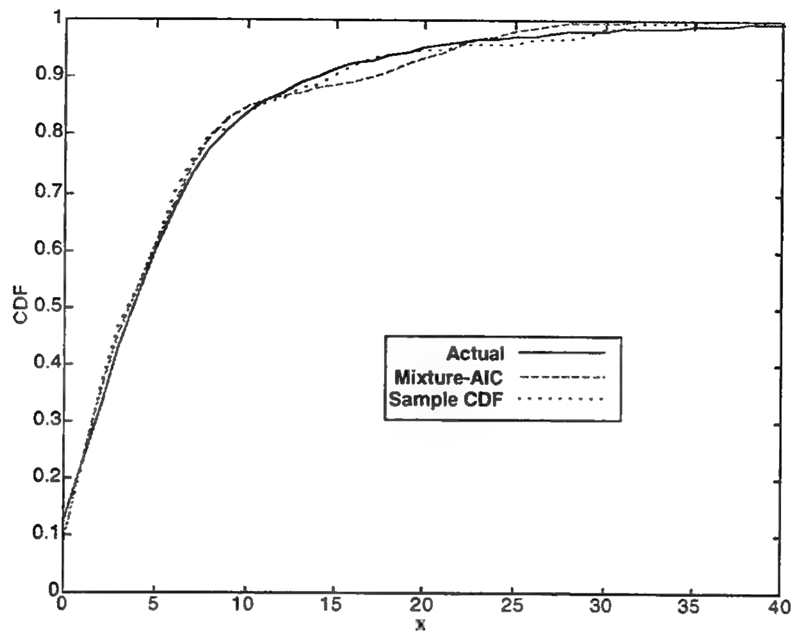

Figure 3: Actual CDF, Sample CDF, and Mixture-Model-Based AIC CDF Estimate of Example 2 With $n = 200$ Observations

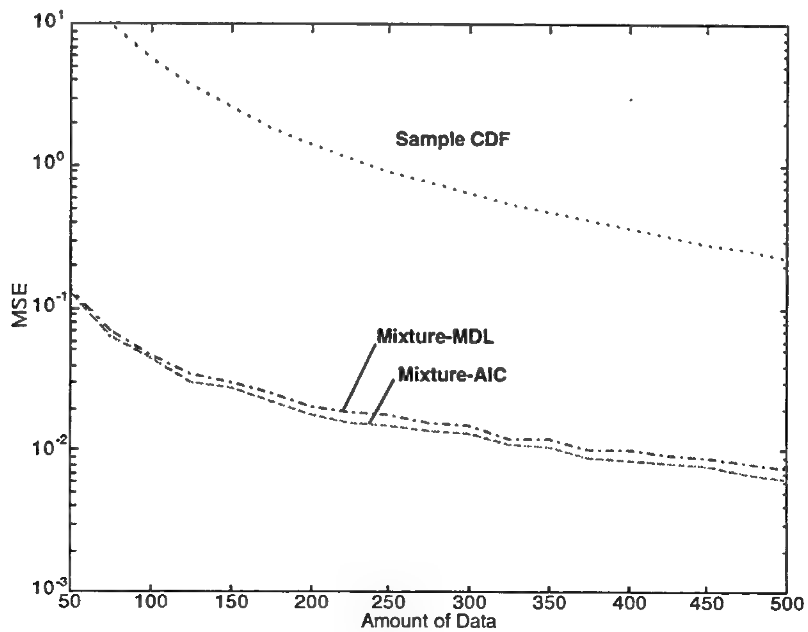

Figure 4: MSE for Sample CDF and Mixture-Model-Based CDF Estimate Using AIC and MDL Model Orders for Example 2

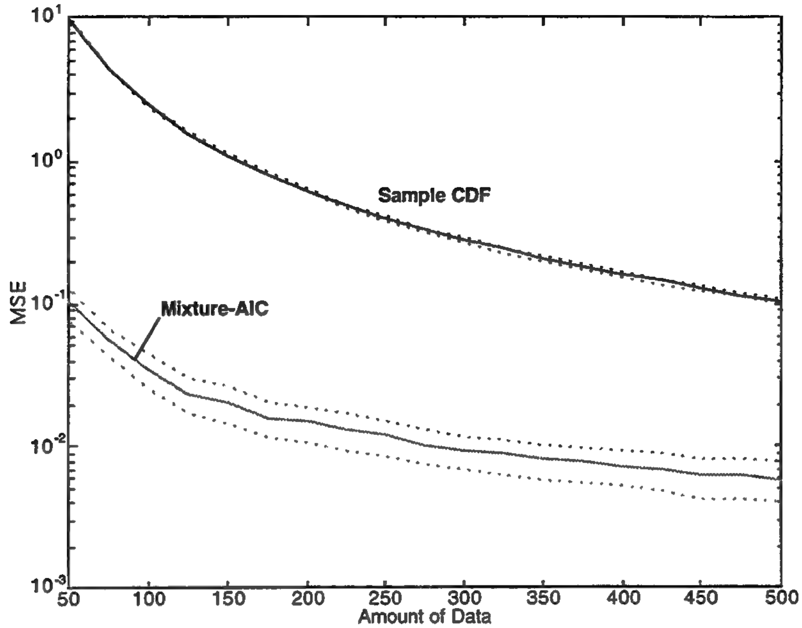

Figure 5: MSE Estimate With Three Standard Deviation Confidence Bounds for Sample CDF and Mixture-Model-Based AIC CDF Estimate of Example 1

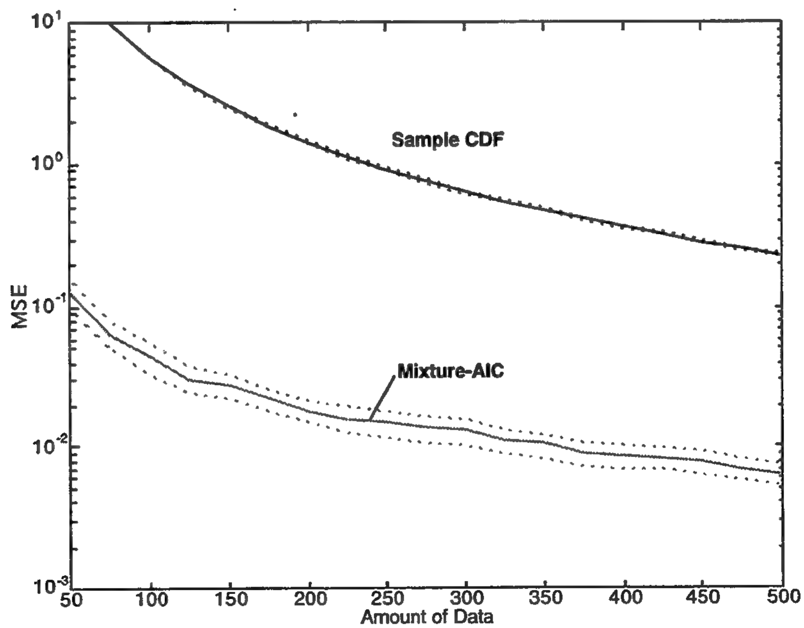

Figure 6: MSE Estimate With Three Standard Deviation Confidence Bounds for Sample CDF and Mixture-Model-Based AIC CDF Estimate of Example 2

PAGE TEST DETECTION PROBABILITY COMPARISON

In this section, the Brownian motion and moment-matching approximations of Han, Willett, and Abraham [4], the asymptotic and fixed sample size methods of Broder [1], and the Poisson mixture method are compared to the sample CDF for determining the Page test probability of detection for three common signal types. The locally optimal nonlinearity, $r(z)$, for each of three signal types is submitted to the Page test with a bias chosen to optimize the asymptotic Page test performance for the specified signal-to-noise ratio (SNR), s, as described in [11]. The Brownian motion, moment-matching, and asymptotic approximations require the mean and variance of the Page test update when a signal is present. These parameters, the locally optimal nonlinearity, and other pertinent signal information are found in table 3. Here, $\mathcal{N}(\mu, \sigma^2)$ represents a Gaussian random variable with mean μ and variance σ^2, $\mathrm{Exp}(\lambda)$ represents an exponential random variable with mean λ, and $\mathcal{X}_n^2(\delta)$ represents a noncentral chi-squared random variable with n degrees of freedom and noncentrality parameter δ.

Table 3: Pertinent Signal Information

Type	$r(z)$	Bias τ_s	$\mathrm{E}_s[r(z)]$	$\mathrm{Var}_s[r(z)]$
$\mathcal{N}(\sqrt{s}, 1)$	$\sqrt{s}z - \tau_s$	$\frac{s}{2}$	$\frac{s}{2}$	s
$\mathrm{Exp}(1+s)$	$\frac{s}{s+1}z - \tau_s$	$\log(1+s)$	$s - \tau_s$	s^2
$\mathcal{X}_2^2(s)$	$z - \tau_s$	$2\left(1+\frac{2}{s}\right)\log\left(1+\frac{s}{2}\right)$	$s + 2 - \tau_s$	$4(s+1)$

The data observations used in the sample CDF generation and the Poisson mixture model approximation are stopping times for the Page test. They are generated according to

$$x_i = \inf\{k \geq 0 : W_k \geq h\} , \tag{22}$$

where

$$W_k = \max\{0, W_{k-1} + r(z_k)\} , \tag{23}$$

$W_0 = 0$, and the $\{z_k\}$ are distributed as described in the first column of table 3. This is done for $i = 1, \ldots, n$, with independent observations of the $\{z_k\}$ sequence. The threshold for all simulations was arbitrarily set to $h = 10$.

CDFs for two SNR values as a function of signal duration and the MSE as a function of SNR between the sample CDF and the Poisson mixture method, Brownian motion, moment-matching, asymptotic, and fixed sample size test approximations are determined for each of the signal types. These results are found in figures 7 and 8 for the Gaussian shift in mean signal, in figures 9 and 10 for the exponential signal, and in figures 11 and 12 for the noncentral chi-squared signal. The sample CDF is generated using 2000 data observations (independent of those used in the mixture model estimation) and is considered to be representative of the actual Page test probability of detection. The Poisson mixture method was allotted n = 100, 250, and 500 data observations to estimate the mixture parameters with the stopping tolerances

$$TOL_1 = 10^{-4} , \tag{24}$$

$$TOL_2 = 10^{-2} , \tag{25}$$

and

$$TOL_3 = 10^{-3} . \tag{26}$$

The values for these tolerances were chosen after simulation results indicated minimal improvement in the MSE between the actual and estimated CDF for smaller values. The MSE plots for the Poisson mixture method of figures 8, 10, and 12 are the averages of 50 samples of the squared error between the sample CDF (computed once with 2000 data samples) and the Poisson mixture approximation CDF (computed 50 times using the AIC-based model order). The squared error is the term inside the expectation in equation (20), with $X_{\max} = 100$, $F(x\,|\phi, \alpha)$ as the estimated CDF, and $F(x)$ as the sample CDF.

Note that, as expected, the Brownian motion provides inadequate approximation for the non-Gaussian signals, and, for the Gaussian signal, overestimates the detection probability as a function of duration because the *continuous-time* model can regulate and declare a detection at any time, whereas the Page test cannot. The moment-matching technique provided excellent performance for the Gaussian signal but performed poorly on the exponential and noncentral chi-squared signals. The Poisson mixture model method provides the best

approximation of Page test detection probability for all three signal types as long as enough data samples ($n = 500$) are used in the estimation of the model parameters. The method based on the asymptotic distribution of the stopping time and the lower bound based on the fixed sample size test are seen to provide the worst overall performance.

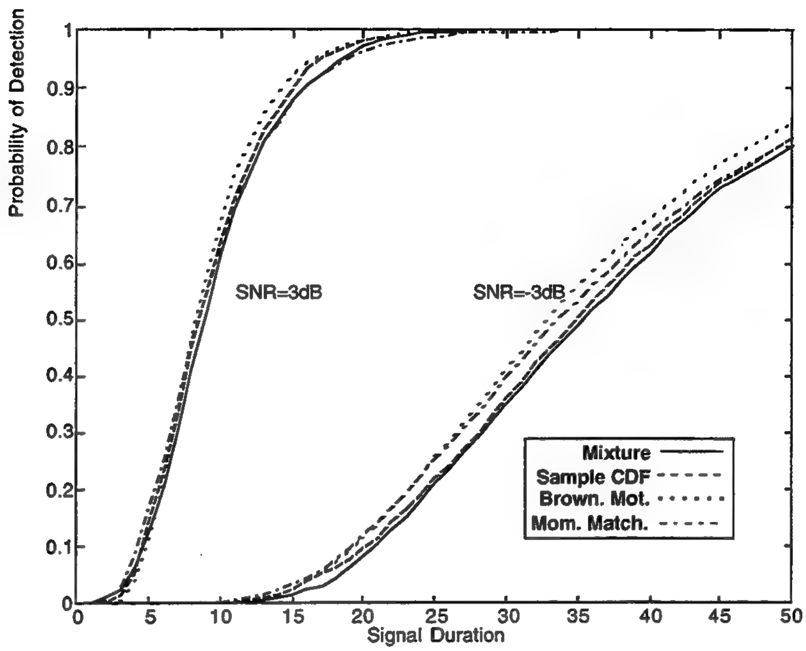

Figure 7: Page Test Probability of Detection for a Gaussian Signal via Brownian Motion Approximation, Moment Matching Method, Sample CDF of 2000 Observations, and Poisson Mixture Approximation From 500 Observations

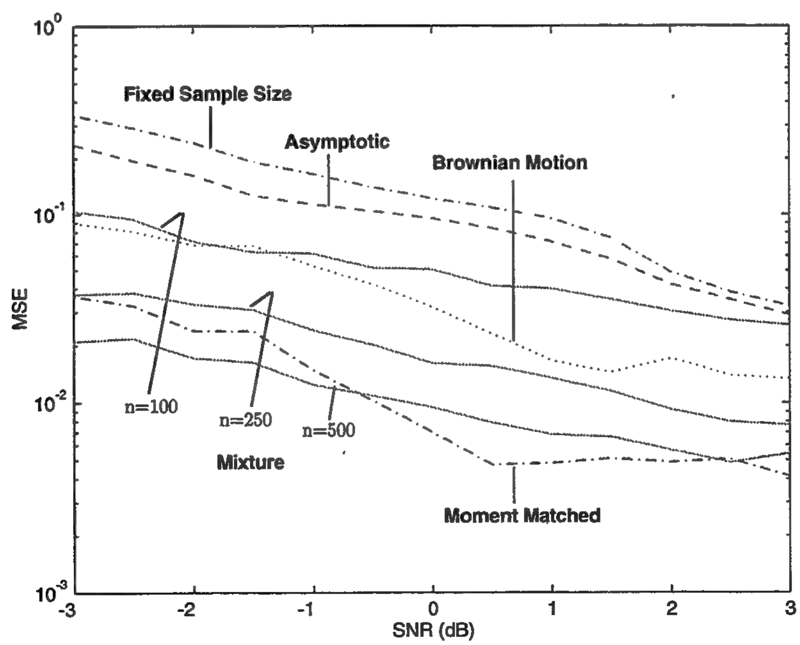

Figure 8: MSE as a Function of SNR for Poisson Mixture, Brownian Motion, Moment-Matching, Asymptotic, and Fixed Sample Size Approximations for a Gaussian Signal

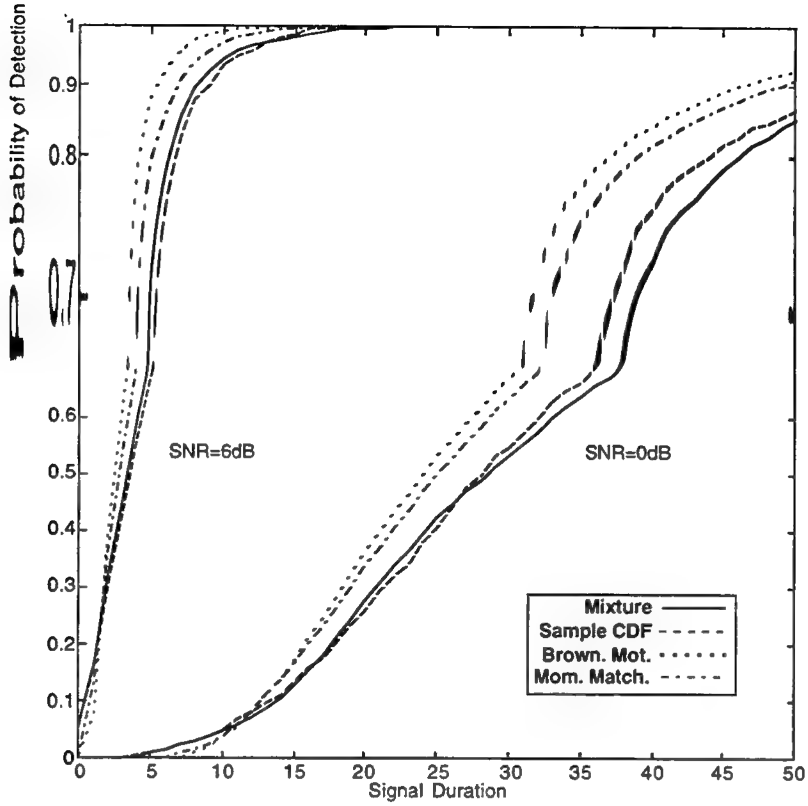

Figure 9: Page Test Probability of Detection for an Exponential Signal via Brownian Motion Approximation, Moment-Matching Method, Sample CDF of 2000 Observations, and Poisson Mixture Approximation From 500 Observations

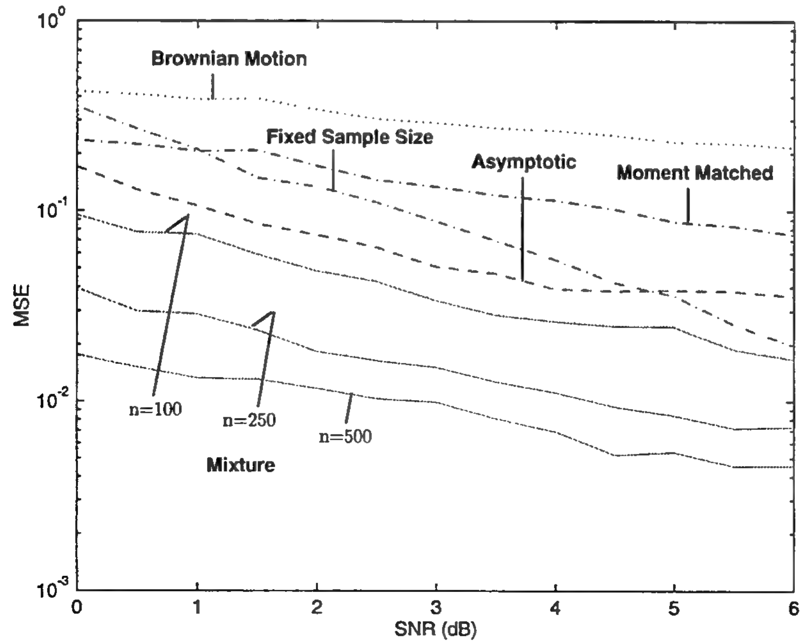

Figure 10: MSE as a Function of SNR for Poisson Mixture, Brownian Motion, Moment-Matching, Asymptotic, and Fixed Sample Size Approximations for an Exponential Signal

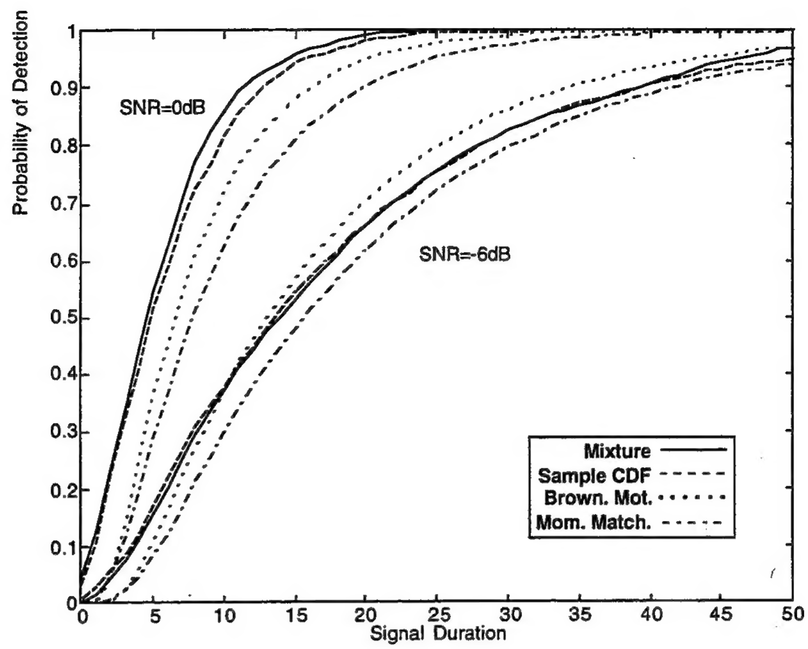

Figure 11: Page Test Probability of Detection for a Noncentral Chi-Squared Signal via Brownian Motion Approximation, Moment-Matching Method, Sample CDF of 2000 Observations, and Poisson Mixture Approximation From 500 Observations

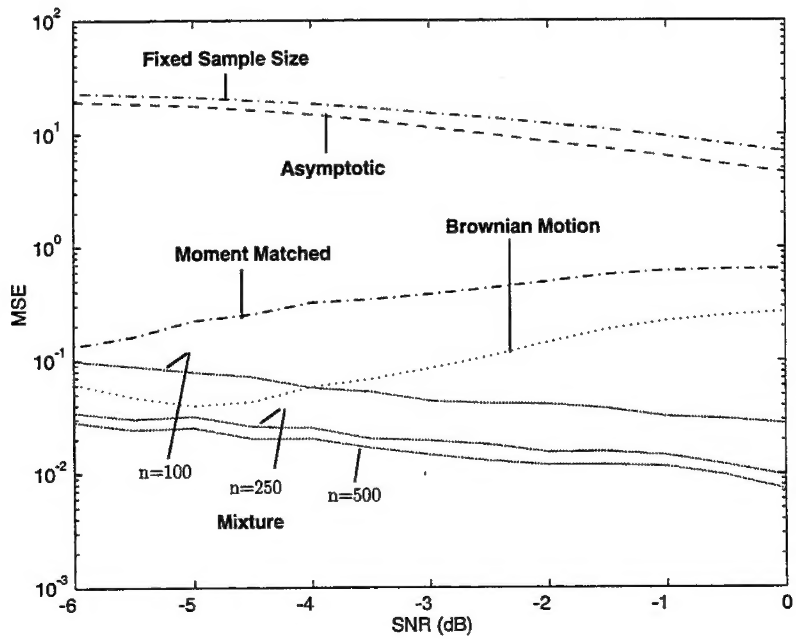

Figure 12: MSE as a Function of SNR for Poisson Mixture, Brownian Motion, Moment-Matching, Asymptotic, and Fixed Sample Size Approximations for a Noncentral Chi-Squared Signal

CONCLUSIONS

The approximation of the probability of detecting a finite duration signal with the Page test through the use of a mixture of Poisson random variables has been explored. The EM algorithm has been utilized to determine the MLEs of the proportion and mean parameters of the individual Poisson random variables in a mixture of a specified order. The AIC and Rissanen's MDL were considered for choosing the best model order based on the data, with Akaike's method providing slightly better results. The MSE between the CDF of the estimated Poisson mixture and mixtures of either Poisson random variables (i.e., the data is correctly modeled) or of Poisson and geometric random variables (i.e., the data is not necessarily correctly modeled) was seen to decrease as the amount of data used to estimate the mixture parameters increased. The dominance of the Poisson mixture method over the sample CDF was demonstrated for these two types of mixtures.

The Page test probability of detection was evaluated for Gaussian, exponential, and noncentral chi-squared signals using the Brownian motion and moment-matching approximations of Han, Willett, and Abraham [4], the asymptotic and fixed sample size methods of Broder [1], and the proposed Poisson mixture model method and was then compared to the sample CDF. The Poisson mixture method utilizing the AIC for model order choice yielded the best performance provided enough data were used to estimate the mixture parameters. The Brownian motion and moment-matching approximations were seen to deteriorate for non-Gaussian signal types and for weak Gaussian signals. However, when the Page test update is Gaussian, the moment-matching method of [4] provides adequate performance with minimal computational effort.

REFERENCES

[1] B. Broder, *Quickest Detection Procedures and Transient Signal Detection.* Ph.D thesis, Princeton University, Princeton, NJ, 1990.

[2] D. A. Abraham, "Sequential vs. Sliding Fixed Block Detection of Finite Unknown Duration Signals," in *Proceedings of The Technical Cooperation Program, Twenty-Third Annual International Meeting*, Auckand, New Zealand, November 1994.

[3] E. S. Page, "Continuous Inspection Schemes," *Biometrika*, vol. 41, 1954, pp. 100–114.

[4] C. Han, P. K. Willett, and D. A. Abraham, "Some Bounds and Approximations to the Probability of Detection of a Transient Signal," Technical Report 94-4, University of Connecticut, Storrs, CT, October 1994.

[5] R. A. Khan, "A Note on Page's Two-sided Cumulative Sum Procedure," *Biometrika*, vol. 68, no. 3, 1981, pp. 717–719.

[6] S. M. Ross, *Introduction to Probability Models*, 5th ed., Academic Press, Inc., NY, 1993.

[7] A. P. Dempster, N. M. Laird, and D. B. Rubin, "Maximum Likelihood From Incomplete Data Via the EM Algorithm," *Journal of the Royal Statistical Society, Series B*, vol. 2, 1977, pp. 1–38.

[8] R. A. Redner and H. F. Walker, "Mixture Densities, Maximum Likelihood and the EM Algorithm," *SIAM Review*, vol. 26, no. 2, April 1984.

[9] H. Akaike, "A New Look at the Statistical Model Identification Problem," *IEEE Transactions on Automatic Control*, December 1974, pp. 716–723.

[10] J. Rissanen, "Modeling by Shortest Data Description," *Automatica*, September 1978, pp. 465–471.

[11] D. A. Abraham, "Asymptotically Optimal Bias for a General Non-linearity in Page's Test," *IEEE Transactions on Aerospace and Electronic Systems*, 1995 (To be published, July 1995).

INITIAL DISTRIBUTION LIST

Addressee	No. of Copies
Coast Guard Academy, New London, CT	
J. Wolcin	1
Naval Undersea Warfare Center Detachment, West Palm Beach	
R. Kennedy	1
Office of Naval Research, Code 451	
T. G. Goldsberry	1
N. Harned	1
M. Shipley	1
The Pennsylvania State University	
R. Young	1
Program Executive Officer USW ASTO	
J. Polcari	1
Princeton University	
S. Schwartz	1
Space and Naval Warfare Systems Command	
R. Holland	1
University of Connecticut	
Y. Bar Shalom	1
D. Dey	1
N. Mukhopadhyay	1
P. Willett	1
University of Rhode Island	
D. Tufts	1
Defense Technical Information Center	12